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PARAMETRIC COEFFICIENTS
IN THE
DIFFERENTIAL GEOMETRY OF CURVES.

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INTRODUCTION.

The writer of the following pages was led to the investigations on parametric coefficients which they contain, from suggestions arising from the following six papers. The first three have been published in the Journal of the Asiatic Society of Bengal, the next two in the Bulletin of the Calcutta Mathematical Society, and the last one, which is yet unpublished, obtained the Griffiths' Prize of the Calcutta University last year.

1. A General Theory of Osculating Conics. (Journal, A. S. B., Vol. IV, No. 4, New Series.)
2. Geometrical Theory of a Plane Non-Cyclic Arc, Finite as well as Infinitesimal. (Journal, A. S. B., Vol. IV, No. 8, New Series.)
3. A General Theory of Osculating Conics, 2nd Paper. (Journal, A. S. B., Vol. IV, No. 10, New Series.)
4. New Methods in the Geometry of a Plane Arc. (Bulletin, C. M. S., Vol. I, No. 1.)
5. On Rates of Variation of the Osculating Conic. (Bulletin, C. M. S., Vol. I, No. 2.)
6. On the Infinitesimal Analysis of an Arc. (Griffiths' Prize Essay, 1909.)

In paper No. 2, approximate expressions for the radius of a circle through any three points of an arc, the difference between an arc and its chord and the area of the segment enclosed between them, etc., were obtained in terms of arc length, radius of curvature and aberrancy, by elementary geometry. As complete expressions could not be obtained by geometrical methods and as the writer was not aware of any existing general method by which such problems could be solved, he was led to devise the method of 'Parametric Coefficients', of which a first sketch has been given in paper No. 6.

In paper No. 3, certain expressions in differential form naturally arose as coefficients. These divided by proper powers

of dt, are the first few parametric coefficients in two dimensions. As the writer had in view the extension of the methods to osculating cubics, he proceeded with the study of parametric coefficients in general, which eventually also gave a solution of the problems which had arisen in paper No. 2.

Three important series of these parametric coefficients occurred as coefficients when

$$L \equiv (X-x)\ Dx + (Y-y)\ Dy,\ M \equiv (X-x)\ Dy - (Y-y)\ Dx \text{ and } N \equiv (X-x)\ D^2y - (Y-y)\ D^2x,$$

were expanded in ascending powers of the parameter t. This has suggested the name 'Parametric Coefficients'.

It was apparent that a suitable cubic V in M, N would yield an expansion in t, commencing with At^9, so that $V=o$ and $A=o$ would be, respectively, the osculating cubic and the differential equation of the cubic. The result $V=o$ has been worked out in paper No. 6. The coefficients are somewhat lengthy.

The writer naturally sought for a suitable parameter which would simplify the expression V. This parameter was discovered while writing paper No. 5. It is the second intrinsic parameter in two dimensions.

The writer is indebted to the kindness of Professor A. R. Forsyth, F.R.S., of Trinity College, Cambridge, for having supplied him, among other things, with certain references to the modern theory of differential equations, where by use of Lie's transformations, one can deduce the equation of the osculating cubic, as also the general differential equation of the cubic. These are given fully in "Projective Differential Geometry of Curves and Ruled Surfaces", by Wilczynski (Teubner, 1906). The methods are far from elementary and the results are expressed as invariants, which have necessitated further investigations to interpret geometrically. The method of the present writer is elementary in character and the results are expressed in terms of invariants, which have direct geometrical significance. In fact it is a distinct merit, of the method of parametric coefficients, to have achieved, by elementary method, results which have been treated by advanced analysis.

A word may be said on the method by which, for example, the equation of the osculating conicoid has been obtained. Apart from the result, which may be claimed as new, the method is interesting. It is simple and general in its scope. In a first stage, it occurs in papers Nos. 1 and 3, where there is a sort of informal use of the 'transforming factor' and reduction to parametric forms.

The conception of 'intrinsic parameters' is a fundamental one in the theory of curves. Except the first, namely,

the arc-length, the rest do not seem to have attracted the notice of geometricians. Although suggested by paper No. 5, the name and use first occurs in the present paper.

Finally, the writer must acknowledge his great indebtedness to Professor C. E. Cullis, M.A., Ph.D., for kindly encouragement and many suggestions.

I. General Conceptions of Parametric Coefficients.

I. Parametric Coefficients defined in n-dimensional space.

Let a curve in n-dimensional space be defined by

$$x_1 = F_1(t),\ x_2 = F_2(t),\ x_3 = F_3(t), \ldots\ldots\ldots\ldots,\ x_n = F_n(t),$$

where $x_1, x_2, x_3 \ldots\ldots\ldots, x_n$ are the co-ordinates of a point P, on the curve, with reference to n axes mutually orthogonal, through a given origin. Any of the co-ordinates x is the length of perpendicular, with proper sign, on the $(n-1)$ dimensional space passing through the remaining $(n-1)$ axes. The functions $F_1, F_2, F_3, \ldots\ldots\ldots, F_n$, as also their derivatives, up to any required order, are supposed to be uniform, finite and continuous, within the limits of value considered of the parameter t.

If we write

$$\frac{d^r}{dt^r} \equiv D^r, \text{ where } r \text{ is a positive integer,}$$

and

$$\Sigma\, D^{m_1} x_r\, D^{m_2} x_r \equiv (m_1, m_2), \text{ where } r = 1, 2, 3, \ldots, n, \text{ then } (m_1, m_2)$$

will be called a parametric coefficient of class 1.

Again, if we write

$$\Sigma \begin{vmatrix} D^{m_1} x_{r_1}, & D^{m_2} x_{r_1} \\ D^{m_1} x_{r_2}, & D^{m_2} x_{r_2} \end{vmatrix}^2 \equiv [m_1, m_2]^2$$

where $r_1 = 1, 2, 3, \ldots, n$

$r_2 = 1, 2, 3, \ldots, n$

then $[m_1, m_2]$ will be called a parametric coefficient of class 2.

Similarly, if we write

$$\Sigma \begin{vmatrix} D^{m_1} x_{r_1}, & D^{m_2} x_{r_1}, & D^{m_3} x_{r_1} \\ D^{m_1} x_{r_2}, & D^{m_2} x_{r_2}, & D^{m_3} x_{r_2} \\ D^{m_1} x_{r_3}, & D^{m_2} x_{r_3}, & D^{m_3} x_{r_3} \end{vmatrix}^2 \equiv [m_1, m_2, m_3]^2$$

where $r_1 = 1, 2, 3, \ldots\ldots\ldots\ldots, n,$
$r_2 = 1, 2, 3, \ldots\ldots\ldots\ldots, n,$
$r_3 = 1, 2, 3, \ldots\ldots\ldots\ldots, n,$

then $[m_1, m_2, m_3]$ will be called a parametric coefficient of class 3, and so on.

Finally

$$\begin{vmatrix} D^{m_1}x_1, \ldots, D^{m_n}x_1 \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ D^{m_1}x_n, \ldots, D^{m_n}x_n \end{vmatrix} \equiv [m_1, m_2, \ldots, m_n]$$

will be called a parametric coefficient of class n, or of the highest class, for the curve in n-dimensional space.

Further

$$\Sigma \begin{vmatrix} D^{m_1}x_{r_1}, & D^{m_2}x_{r_1} \\ D^{m_1}x_{r_2}, & D^{m_2}x_{r_2} \end{vmatrix} \begin{vmatrix} D^{p_1}x_{r_1}, & D^{p_2}x_{r_1} \\ D^{p_1}x_{r_2}, & D^{p_2}x_{r_2} \end{vmatrix} \equiv [m_1, m_2 \mid p_1, p_2]$$

where

$r_1 = 1, 2, 3, \ldots\ldots n$
$r_2 = 1, 2, 3, \ldots\ldots n$

will be called the moment of the parametric coefficients

$$[m_1, m_2] \text{ and } [p_1, p_2].$$

Similarly,

$$\Sigma \begin{vmatrix} D^{m_1}x_{r_1}, & D^{m_2}x_{r_1}, & D^{m_3}x_{r_1} \\ D^{m_1}x_{r_2}, & D^{m_2}x_{r_2}, & D^{m_3}x_{r_2} \\ D^{m_1}x_{r_3}, & D^{m_2}x_{r_3}, & D^{m_3}x_{r_3} \end{vmatrix} \begin{vmatrix} D^{p_1}x_{r_1}, & D^{p_2}x_{r_1}, & D^{p_3}x_{r_1} \\ D^{p_1}x_{r_2}, & D^{p_2}x_{r_2}, & D^{p_3}x_{r_2} \\ D^{p_1}x_{r_3}, & D^{p_2}x_{r_3}, & D^{p_3}x_{r_3} \end{vmatrix}$$
$$\equiv [m_1, m_2, m_3 \mid p_1, p_2, p_3]$$

where

$r_1 = 1, 2, 3, \ldots\ldots n$
$r_2 = 1, 2, 3, \ldots\ldots n$
$r_3 = 1, 2, 3, \ldots\ldots n$

will be called the moment of the parametric coefficients

$[m_1, m_2, m_3]$ and $[p_1, p_2, p_3]$, and so on.

We have thus moments of parametric coefficients, taken in pairs, of any class, from the second upwards. If the coefficients be identical, then their moment is equal to their product.

Thus,

$$[m_1, m_2 \mid m_1, m_2] = [m_1, m_2]^2$$
$$[m_1, m_2\ m_3 \mid m_1, m_2, m_3] = [m_1, m_2, m_3,]^2$$

and so on.

Also the moment of two parametric coefficients of the highest class n, is the product of the coefficients.

Thus, $[m_1, m_2, \ldots . m_n \mid p_1, p_2, \ldots . p_n]$
$= [m_1, m_2, \ldots . , m_n][p_1, p_2, \ldots . , p_n]$

These moments are expressible as simple determinant functions of parametric coefficients of class 1.

Thus,

$$[m_1, m_2 \mid p_1, p_2] = \begin{vmatrix} (m_1, p_1), & (m_2, p_1) \\ (m_1, p_2), & (m_2, p_2) \end{vmatrix}$$

$$[m_1, m_2, m_3 \mid p_1, p_2, p_3] = \begin{vmatrix} (m_1, p_1), & (m_2, p_1), & (m_3, p_1) \\ (m_1, p_2), & (m_2, p_2), & (m_3, p_2) \\ (m_1, p_3), & (m_2, p_3), & (m_3, p_3) \end{vmatrix}$$

and so on.

Again, since for space of n dimensions, parametric coefficients of class higher than n must vanish, we have

$$[m_1, m_2, \ldots . m_n, m_{n+1}] = 0, \ [p_1, p_2, \ldots . p_n, p_{n+1}] = 0$$

and therefore

$$\begin{vmatrix} (m_1, p_1), & \ldots\ldots\ldots\ldots\ldots, & (m_{n+1}, p_1) \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ (m_1, p_{n+1}), & \ldots\ldots\ldots\ldots, & (m_{n+1}, p_{n+1}) \end{vmatrix} = 0$$

Note.—The usefulness of parametric coefficients first suggested itself to the writer while studying the properties of osculating conics. In a paper on Osculating Conics (second paper), published in the Journal, Asiatic Society of Bengal, Vol. IV, No. 10 (New series), eight of these coefficients, which were called P, Q, Q_1, R, S, R', S', T, came out in the general differential equation of the osculating conic and other expressions connected with the osculating conic. A more extended use of these coefficients was made in paper No. 6. The notation adopted in that paper was the old notation strengthened by additional letters of the alphabet and additional dashes. The writer is indebted to the kind suggestion of Professor C. E. Cullis, M.A., Ph.D., for an improved notation. The notation adopted in this paper is really the outcome of this suggestion and of the necessity to suit n-dimensional curves. The name parametric coefficients has been first adopted in this paper.

2. The n Intrinsic Parameters of a curve in n-dimensional space and their geometrical interpretations.

The first Intrinsic Parameter of a curve in n-dimensional space may be defined as

$$s_1 \equiv \int_{t_o}^{t} (1,1)^{\frac{1}{2}} dt$$

The second Intrinsic Parameter may be defined as

$$s_2 \equiv \int_{t_o}^{t} [1,\ 2]^{\frac{1}{3}} dt$$

The third Intrinsic Parameter may be defined as

$$s_3 \equiv \int_{o}^{t} [1,2,3]^{\frac{1}{6}} dt$$

The n^{th} Intrinsic Parameter may be defined as

$$s_n \equiv \int_{t_o}^{t} [1,\ 2, \ldots n]^{\frac{2}{n(n+1)}} dt$$

A plane curve has evidently only two Intrinsic Parameters s_1 and s_2, and a curve in space only three, s_1, s_2 and s_3.

Let P_o and P be any two points on the curve, corresponding to given value t_o and t of the parameter t. Take a large number N, of consecutive points on the curve, from P_o to P, corresponding to N equal small increments δt of t, so that $N\delta t = t - t_o$. Then if $(x_1, x_2, \ldots\ldots, x_n)$ and $(x_1 + \delta x_1, x_2 + \delta x_2, \ldots\ldots, x_n + \delta x_n)$ be the co-ordinates of any two consecutive points P_r, P_{r+1}, the length of the chord $P_r P_{r+1}$ is

$$\{(\delta x_1)^2 + (\delta x_2)^2 + \ldots . + (\delta x_n)^2\}^{\frac{1}{2}}$$

and the sum of the lengths of N such chords is

$$\Sigma \{(\delta x_1)^2 + (\delta x_2)^2 + \ldots . + (\delta x_n)^2\}^{\frac{1}{2}}$$

$$= \Sigma \left\{ \left(\frac{\delta x_1}{\delta t} \right)^2 + \left(\frac{\delta x_2}{\delta t} \right)^2 + \ldots . + \left(\frac{\delta x_n}{\delta t} \right)^2 \right\}^{\frac{1}{2}} \delta t$$

This sum has a limiting value, when N is infinitely large, which may be written as

$$\int_{t_o}^{t} (1,\ 1)^{\frac{1}{2}} dt$$

and which is therefore the first Intrinsic Parameter (s_1). The first intrinsic parameter is evidently the same as arc-length (s).

Again, if P_r, P_{r+1}, P_{r+2} be any three consecutive points on the curve corresponding to equal small increments δt of t, then if $x_1, x_2, \ldots, x_n$ be the co-ordinates of P_r, those of P_{r+1} and P_{r+2} will be

$$x_1+\delta x_1,\ x_2+\delta x_2, \ldots\ldots\ldots\ldots\ldots\ldots, x_n+\delta x_n,$$

and $\quad x_1+2\delta x_1+\delta^2 x_1,\ x_2+2\delta x_2+\delta^2 x_2, \ldots\ldots, x_n+2\delta x_n+\delta^2 x_n,$

respectively.

The projection δS_{12} of the area δS of the triangle $P_r\,P_{r+1}\,P_{r+2}$ on the (x_1, x_2) plane is

$$\frac{1}{2!}\begin{vmatrix} 1, & x_1, & x_2, \\ 1, & x_1+\delta x_1, & x_2+\delta x_2 \\ 1, & x_1+2\delta x_1+\delta^2 x_1, & x_2+2\delta x_2+\delta^2 x_2 \end{vmatrix}$$

$$=\frac{1}{2!}\begin{vmatrix} 1, & 0\ , & 0 \\ 1, & \delta x_1\ , & \delta x_2 \\ -1, & \delta^2 x_1, & \delta^2 x_2 \end{vmatrix}$$

$$=\frac{1}{2!}(\delta x_1 \delta^2 x_2-\delta x_2 \delta^2 x_1)$$

or,

$$\delta S_{12}=\frac{1}{2!}\left(\frac{\delta x_1}{\delta t}\ \frac{\delta^2 x_2}{\delta t^2}-\frac{\delta x_2}{\delta t}\ \frac{\delta^2 x_1}{\delta t^2}\right)\delta t^3$$

But $\quad (\delta S)^2=\Sigma(\delta S_{12})^2=\left(\frac{1}{2!}\right)^2\Sigma\left(\frac{\delta x_1}{\delta t}\ \frac{\delta^2 x_2}{\delta t^2}-\frac{\delta^2 x_1}{\delta t^2}\ \frac{\delta x_2}{\delta t}\right)^2\delta t^6$

Therefore

$$\int_{P_0}^{P}(\delta S)^{\frac{1}{3}}=\frac{1}{(2!)^{\frac{1}{3}}}\int_{t_0}^{t}\left\{\Sigma\left(\frac{dx_1}{dt}\ \frac{d^2x_2}{dt^2}-\frac{d^2x_1}{dt^2}\ \frac{dx_2}{dt}\right)^2\right\}^{\frac{1}{6}}dt$$

$$=\frac{1}{(2!)^{\frac{1}{3}}}\int_{t_0}^{t}[1,\ 2]^{\frac{1}{3}}dt.$$

So that the second Intrinsic Parameter

$$s_2=\int_{t_0}^{t}[1,\ 2]^{\frac{1}{3}}dt=(2!)^{\frac{1}{3}}\int_{P_0}^{P}(\delta S)^{\frac{1}{3}}$$

Similarly, if P_r, P_{r+1}, P_{r+2}, P_{r+3} be any four consecutive points on the curve, corresponding to equal small increments δt

of t, then if $x_1, x_2, \ldots\ldots, x_n$ be the co-ordinates of P_r those of $P_{r+1}, P_{r+2}, P_{r+3}$ will be respectively

$$\begin{gathered}
x_1 + \delta x_1, \qquad x_2 + \delta x_2, \ldots\ldots\ldots, x_n + \delta x_n\,, \\
x_1 + 2\delta x_1 + \delta^2 x_1, \quad x_2 + 2\delta x_2 + \delta^2 x_2, \ldots x_n + 2\delta x_n + \delta^2 x_n\,, \\
x_1 + 3\delta x_1 + 3\delta^2 x_1 + \delta^3 x_1, \; x_2 + 3\delta x_2 + 3\delta^2 x_2 + \delta_3 x_2, \ldots x_n + 3\delta x_n + 3\delta^2 x_n \\
+ \delta^3 x_n\,.
\end{gathered}$$

The projection δV_{123} of the volume δV of the tetrahedron $P_r\, P_{r+1}\, P_{r+2}\, P_{r+3}$ on the space (x_1, x_2, x_3) is

$$\frac{1}{3!}\begin{vmatrix}
1, & x_1, & x_2, & x_3 \\
1, & x_1 + \delta x_1, & x_2 + \delta x_2, & x_3 + \delta x_3 \\
1, & x_1 + 2\delta x_1 + \delta^2 x_1, & x_2 + 2\delta x_2 + \delta^2 x_2, & x_3 + 2\delta x_3 + \delta^2 x_3 \\
1, & x_1 + 3\delta x_1 + 3\delta^2 x_1 + \delta^3 x_1, & x_2 + 3\delta x_2 + 3\delta^2 x_2 + \delta_3 x_2, & x_3 + 3\delta x_3 + 3\delta^2 x_3 + \delta^3 x_3
\end{vmatrix} = \frac{1}{3!}\begin{vmatrix}
\delta x_1, & \delta x_2, & \delta x_3 \\
\delta^2 x_1, & \delta^2 x_2, & \delta^2 x_3 \\
\delta^3 x_1, & \delta^3 x_2, & \delta^3 x_3
\end{vmatrix}$$

$$\text{But } (\delta V)^2 = \Sigma(\delta V_{123})^2 = \frac{1}{(3!)^2}\;\Sigma\begin{vmatrix}
\dfrac{\delta x_1}{\delta t}, & \dfrac{\delta x_2}{\delta t}, & \dfrac{\delta x_3}{\delta t} \\
\dfrac{\delta^2 x_1}{\delta t^2}, & \dfrac{\delta^2 x_2}{\delta t^2}, & \dfrac{\delta^2 x_3}{\delta t^2} \\
\dfrac{\delta^3 x_1}{\delta t^3}, & \dfrac{\delta^3 x_2}{\delta t^3}, & \dfrac{\delta^3 x_3}{\delta t^3}
\end{vmatrix}^2 (\delta t)^{12}$$

Therefore

$$\int_{P_0}^{P} (\delta V)^{\frac{1}{6}} = \frac{1}{(3\,!)^{\frac{1}{6}}}\int_{t_0}^{t} \left\{ \Sigma[1,\,2,\,3]^2 \right\}^{\frac{1}{12}} dt$$

$$= \frac{1}{(3\,!)^{\frac{1}{6}}}\int_{t_0}^{t} [1,\,2,\,3]^{\frac{1}{6}} dt$$

So that the third Intrinsic Parameter

$$s_3 = (3!)^{\frac{1}{6}}\int_{P_0}^{P} (\delta V)^{\frac{1}{6}}$$

In the same way if we take $p+1$ consecutive points P_r, P_{r+1},

$P_{r+2}, \ldots\ldots, P_{r+p}$ on the curve, the p-dimensional content of rectilineal figure formed by them is determined by

$$(\delta U_p)^2 = \frac{1}{(p\,!)^2}\Sigma \begin{vmatrix} \delta x_r, \ldots\ldots\ldots, \delta x_{r+p} \\ \delta^2 x_r, \ldots\ldots\ldots, \delta^2 x_{r+p} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \delta^p x_r, \ldots\ldots, \delta^p x_{r+p} \end{vmatrix}^2$$

Therefore

$$\int_{P_o}^{P} (\delta U_p)^{\frac{2}{p(p+1)}} = (p\,!)^{-\frac{2}{p(p+1)}} \int_{t_o}^{t} \{\Sigma[1, 2, \ldots . p]^2\}\, dt^{\frac{1}{p(p+1)}}$$

$$= (p\,!)^{-\frac{2}{p(p+1)}} \int_{t_o}^{t} [1, 2, \ldots . p]^{\frac{2}{p(p+1)}} dt$$

So that if s_p be the p^{th} Intrinsic Parameter

$$s_p = (p\,!)^{\frac{2}{p(p+1)}} \int_{P_o}^{P} (\delta U_p)^{\frac{2}{p(p+1)}}$$

It may not be superfluous here, to point out that the content of a p-dimensional pyramid, whose vertex is P and base the $(p-1)$ dimensional rectilinear figure determined by P', $P'' \ldots, P^{(n)}$ is $\frac{1}{p} U_{p-1} H_{p-1}$ where H_{p-1} is height of vertex over base and U_{p-1} the content of the base, for if we take $\delta\, U_p$ to be the element of U_p between two parallel $(p-1)$ dimensional bases, separated by distance $\delta\, H_{p-1}$, then because U_{p-1} varies as $(H_{p-1})^{p-1}$ we must have

$$U_p = \int_o^{Hp-1} U_{p-1}\, dH_{p-1} = \int_o^{Hp-1} k(H_{p-1})^{p-1} dH_{p-1}$$

$$= \frac{k}{p} H^{p}_{p-1} = \frac{1}{p} H_{p-1}\, U_{p-1}.$$

$$\text{Similarly } U_{p-1} = \frac{1}{p-1} H_{p-2}\, U_{p-2} \text{ and so on.}$$

$$\text{Therefore } U_p = \frac{1}{p!} H_{p-1}\, H_{p-2} \ldots . H_1$$

whence we deduce in the usual way the formula

$$U_p = \frac{1}{p!}\begin{vmatrix} 1, & x_1, & x_2, & \ldots\ldots, & x_p \\ 1, & x_1', & x_2' & \ldots\ldots, & x'_p, \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1, & x_1^{(p)}, & x_2^{(p)}, & \ldots\ldots & x_p^{(p)} \end{vmatrix}$$

where $x_1, x_2, \ldots, x_p$; $x_1', x_2' \ldots x'^{p}; \ldots$; $x_1^{(p)}, x_2^{(p)} \ldots, x_n^{(p)}$; are the co-ordinates of $p+1$ points $P, P', \ldots P^{(p)}$ in p dimensional space.

It will appear from the above geometrical interpretations, that the n parameters $s_1, s_2, \ldots\ldots s_n$ are intrinsically connected with the curve and give, as it were, its measure in respectively one, two,..and n dimensions. The values of s_1, s_2, etc., are independent of the system of axes chosen and of the parameter t and only vary with the positions of P_o and P on the curve. The n co-ordinates of a point P on the curve may be expressed as functions of any one of these parameters. Any $n-1$ independent equations between these intrinsic parameters will determine a curve in n-dimensional space, intrinsically.

Note.—The idea of Intrinsic Parameters was suggested to the writer while investigating rates of variation of the osculating conic (*vide* Bulletin, Calcutta Mathematical Society, Vol. I, No. 2). It was noticed that by introducing the operator $Q^{-\frac{1}{3}}D$, where $Q = DxD^2y - DxD^2x$ results and processes could be remarkably simplified. But $Q^{-\frac{1}{3}}D \equiv \frac{d}{ds_2}$, where $s_2 = \int_{t_o}^{t} Q^{\frac{1}{3}}\, d\,t$ which is the second intrinsic parameter in two dimensions. The extension from two to higher dimensions was natural and easy.

3. The $n-1$ radii of linear curvature of a curve in n dimensional space.

If $s_1, s_2 \ldots\ldots s_n$ be the n intrinsic parameters then

$$\frac{ds_1}{dt} = (1,\,1)^{\frac{1}{2}},\ \frac{ds_2}{dt} = [1,\,2]^{\frac{1}{3}},\ \frac{ds_3}{dt} = [1,\,2,\,3]^{\frac{1}{6}},\ \text{etc.}$$

$$\frac{ds_n}{dt} = [1,\,2,\,3,\ldots n]^{\frac{2}{n(n+1)}}$$

We may define the radius of first curvature as

$$\rho_1 = \left(\frac{ds_1}{ds_2}\right)^3 = \frac{(1,\ 1)^{\frac{3}{2}}}{[1,\ 2]},$$

that of second curvature as

$$\rho_2 = \left(\frac{ds_2}{ds_3}\right)^6 = \frac{[1,\ 2]^2}{[1,\ 2,\ 3]}$$

and that of third curvature as

$$\rho_3 = \left(\frac{ds_3}{ds_4}\right)^{10} = \frac{[1,\ 2,\ 3]^{\frac{5}{3}}}{[1,\ 2,\ 3,\ 4]}$$

and so on.

The radius of $n-1^{th}$ curvature is

$$\rho_{n-1} = \left(\frac{ds_{n-1}}{ds_n}\right)^{\frac{n(n+1)}{2}} = \frac{[1,\ 2,\ 3, \ldots\ldots, n-1]^{\frac{n+1}{n-1}}}{[1,\ 2,\ 3, \ldots\ldots, n]}$$

The dimensions of $\rho_1, \rho_2 \ldots\ldots, \rho_{n-1}$ are obviously of the first degree in length. It is easily shewn that the first radius of curvature represents the radius of a circle through three consecutive points.

Let P, P', P'' be three consecutive points whose co-ordinates are

$$x_1,\ x_2, \ldots . x_n$$
$$x_1 + \delta x_1,\ x_2 + \delta x_2, \ldots x_n + \delta x_n$$
$$x_1 + 2\delta x_1 + \delta^2 x_1,\ x_2 + 2\delta x_2 + \delta^2 x_2, \ldots x_n + 2\delta x_n + \delta^2 x_n$$

The radius ρ of the circle circumscribing $P\ P'\ P''$ is $\dfrac{PP'.\ P'P''.\ P''P}{4\Delta PP'P''}$.

But

$$PP' = \{\Sigma(\delta x)^2\}^{\frac{1}{2}} = (1,\ 1)^{\frac{1}{2}}\ \delta t, \text{ ultimately},$$

$$\text{and } P'P'' = PP',\ P''P = 2\ PP', \text{ ultimately}.$$

Besides

$$4\Delta PP'P'' = 2\{\Sigma(\delta x_1\ \delta^2 x_2 - \delta x_2\ \delta^2 x_1)^2\}^{\frac{1}{2}}$$
$$= 2[1,\ 2]\ \delta t^3, \text{ ultimately}.$$

Therefore

$$\rho = \frac{(1,\ 1)^{\frac{3}{2}}}{[1,\ 2]} = \rho_1.$$

It may be noted that by the use of general parametric coefficients the first radius of curvature has been expressed by the same formula in n dimensions as holds for two dimensions.

This is *à priori* obvious. For, if we take the osculating plane as the plane of (x_1, x_2), then the remaining co-ordinates and their first two derivatives vanish. But the expression $\dfrac{(1, 1)^{\frac{3}{2}}}{[1, 2]}$ is an invariant being the ratio of the differentials of two intrinsic parameters, raised to some power. Therefore it would mean the same thing howsoever we take the co-ordinate axes. So that if it mean the radius of the osculating circle in two variable x_1, x_2 it would mean the same thing in n variables x_1, x_2, $\ldots$, x_n.

The higher intrinsic parameters are expressible in terms of the first intrinsic parameter and the radius of curvature.

For, we have from definitions

$$[1, 2]^{\frac{1}{3}} = \rho_1^{-\frac{1}{3}} (1, 1)^{\frac{1}{2}}, \quad [1, 2, 3]^{\frac{1}{6}} = \rho_2^{-\frac{1}{6}} [1, 2]^{\frac{1}{3}},$$

$$[1, 2, 3, 4]^{\frac{1}{10}} = \rho_3^{-\frac{1}{10}} [1, 2, 3]^{\frac{1}{6}}, \text{ etc.}$$

$$[1, 2, 3, \ldots n]^{\frac{2}{n(n+1)}} = \rho_{n-1}^{-\frac{2}{n(n+1)}} [1, 2, 3 \ldots, n-1]^{\frac{2}{(n-1)n}}.$$

Therefore

$$s_2 = \int_{t_0}^{t} [1, 2]^{\frac{1}{3}} dt = \int_{t_0}^{t} \rho_1^{-\frac{1}{3}} \frac{ds}{dt} \, dt = \int_{s_0}^{s} \rho_1^{-\frac{1}{3}} ds$$

$$s_3 = \int_{t_0}^{t} [1, 2, 3]^{\frac{1}{6}} dt = \int_{s_0}^{s} \rho_1^{-\frac{1}{3}} \rho_2^{-\frac{1}{6}} ds$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$s_n = \int_{t_0}^{t} [1, 2, 3, \ldots n]^{\frac{2}{n(n+1)}} dt$$

$$= \int_{s_0}^{s} \rho_1^{-\frac{1}{3}} \rho_2^{-\frac{1}{6}} \rho_3^{-\frac{1}{10}} \ldots \rho_{n-1}^{-\frac{2}{n(n+1)}} ds.$$

The invariant nature of the intrinsic parameters is, there fore, connected with the invariance of $\rho_1, \rho_2, \rho_3, \ldots \ldots \rho_{n-1}$.

Note.—The conception of the higher radii of curvature as powers of ratios of differentials of two consecutive intrinsic parameters is believed to be new. It introduces a degree of simplicity and uniformity in the study of the higher curvatures.

4. The Sphere and Conicoid of Osculation.

The spheric of $(n-1)$ dimensional boundary which has closest contact with a curve in n dimensional space has for its equation

$$\begin{vmatrix} \Sigma(X_r-x_r)^2, & X_1-x_1, & \ldots\ldots\ldots\ldots\ldots\ldots, & X_n-x_n \\ o, & Dx_1, & \ldots\ldots\ldots\ldots\ldots\ldots, & Dx_n \\ 2\,(1,1), & D^2x_1, & \ldots\ldots\ldots\ldots\ldots\ldots, & D^2x_n \\ 6\,(1,2), & D^3x_1, & \ldots\ldots\ldots\ldots\ldots\ldots, & D^3x_n \\ \cdots & \cdots & \cdots & \cdots \\ n\,(1,n-1)+\dfrac{n(n-1)}{2!}(2,n-1)+\text{etc.}, & D^nx_1, & \ldots, & D^nx_n \end{vmatrix}=0$$

This is deduced from the spheric through $n+1$ points, which has for equation

$$\begin{vmatrix} \Sigma(X_r-x_r)^2, & X_1-x_1, & \ldots\ldots\ldots\ldots\ldots\ldots, & X_n-x_n \\ \Sigma(x_r'-x_r)^2, & x_1'-x_1, & \ldots\ldots\ldots\ldots\ldots\ldots, & x'_n-x_n \\ \cdots & \cdots & \cdots & \cdots \\ \Sigma(x_r^{(n)}-x_r)^2, & x_1^{(n)}-x_1, & \ldots\ldots\ldots\ldots, & x_n^{(n)}-x^n \end{vmatrix}=0$$

or

$$\begin{vmatrix} \Sigma(X_r-x_r)^2, & X_1-x_1, & \ldots\ldots\ldots\ldots\ldots\ldots \\ \Sigma(x_r'-x_r)^2, & x_1'-x_1, & \ldots\ldots\ldots\ldots\ldots\ldots \\ \Sigma\{(x_r''-x_r)^2-2\,(x_r'-x_r)^2\}, & x_1''-2x_1'+x_1, & \ldots\ldots\ldots\ldots \\ \cdots & \cdots & \cdots \\ \Sigma\{(x_r^{(n)}-x_r)^2-n\,(x_r^{(n-1)}-x_r)^2+\text{etc.}\}, & x_1^{(n)}-nx_1^{(n-1)}+\text{etc.}, & \ldots \end{vmatrix}=0$$

or, using the notation of the calculus of Finite Differences

$$\begin{vmatrix} \Sigma(X_r-x_r)^2, & X-x_1, & \ldots\ldots\ldots\ldots\ldots\ldots, & X_n-x_n \\ \Sigma\delta(X_r-x_r)^2_{X_r=x_r}, & \delta x_1, & \ldots\ldots\ldots\ldots\ldots\ldots, & \delta x_n \\ \Sigma\delta^2(X_r-x_r)^2_{X_r=x_r}, & \delta^2 x_1, & \ldots\ldots\ldots\ldots\ldots\ldots, & \delta^2 x_n \\ \cdots & \cdots & \cdots & \cdots \\ \Sigma\delta^n(X_r-x_r)^2_{X_r=x_r}, & \delta^n x_1, & \ldots\ldots\ldots\ldots\ldots\ldots, & \delta^n x_n \end{vmatrix}=0$$

Now if we divide the second, third, etc., rows by δt, $(\delta t)^2$, $(\delta t)^3$, etc., and go to limits, we have

$$Lt\ \frac{\delta^n x_r}{(\delta t)^n}=D^n x_r$$

$$\text{and } Lt\ \frac{\delta^n(X_r-x_r)^2}{(\delta t)^n}_{X_r=x_r}=D^n(X_r-x_r)^2_{X_r=x_r}$$

$$=n\, Dx_r\, D^{n-1}x_r + \frac{n(n-1)}{2!} D^2x_r D^{n-2}x_r + \;\ldots\ldots$$

The equation of the osculating spheric in n-dimensional space can be transformed into

$$\begin{vmatrix} \Sigma(X_r-x_r)^2, & L_1, & L_2, & \ldots\ldots\ldots\ldots\ldots\ldots, & L_n \\ 0 & [1, 2, \ldots n], & 0, & \ldots\ldots\ldots\ldots\ldots\ldots, & 0 \\ 2\,(1,1), & 0 & [1, 2, \ldots n], & \ldots\ldots\ldots\ldots, & 0 \\ 6\,(1,2), & 0, & 0, & \ldots\ldots\ldots\ldots, & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ n\,(1, n-1)+\frac{n(n-1)}{2!}(2, n-2)+\text{etc.}, & 0, & 0, & \ldots, & [1, 2, \ldots n] \end{vmatrix}=0$$

Or,

$$[1, 2, \ldots, n]\Sigma(X_r-x_r)^2-2\,(1,1)L_2-6\,(1,2)\,L_3-\ldots\ldots$$

$$-\left\{n\,(1, n-1)+\frac{n\,(n-1)}{2!}(2, n-1)+\text{etc.}\right\} L_n^2=0.$$

The transforming factor is

$$\begin{vmatrix} 1, & 0, & 0, & \ldots\ldots, & 0 \\ 0, & A_{11}, & A_{12}, & \ldots\ldots, & A_{1n} \\ 0, & A_{21}, & A_{22}, & \ldots\ldots, & A_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0, & A_{n1}, & A_{n2}, & \ldots\ldots, & A_{nn} \end{vmatrix}$$

where A_{11}, A_{12}, ..., A_{nn} are the first minors of the determinant

$$[1, 2, 3, \ldots n] \equiv \begin{vmatrix} Dx_1, & Dx_2, & \ldots\ldots, & Dx_n \\ D^2x_1, & D^2x_2, & \ldots\ldots, & D^2x_n \\ D^3x_1, & D^3x_2, & \ldots\ldots, & D^3x_n \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ D^nx_1, & D^nx_2, & \ldots\ldots, & D^nx_n \end{vmatrix}$$

and

$$L_1 \equiv (X_1-x_1)\,A_{11}+(X_2-x_2)\,A_{12}+\ldots\ldots+(X_n-x_n)\,A_{1n}$$
$$L_2 \equiv (X_1-x_1)\,A_{21}+(X_2-x_2)\,A_{22}+\ldots\ldots+(X_n-x_n)\,A_{2n}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$L_n \equiv (X_1-x_1)\,A_{n1}+(X_2-x_2)\,A_{n2}+\ldots\ldots+(X_n-x_n)\,A_{nn}.$$

In particular the equation of the sphere of closest contact in three dimensional space has for equation

$$[1, 2, 3]\Sigma(X-x)^2-2\,(1,1)\,L_2-6\,(1,2)\,L_3=0$$

where

$$L_2 \equiv -(X-x)[1,3]_{yz} - (Y-y)[1,3]_{zx} - (Z-z)[1,3]_{xy}$$

$$L_3 \equiv \;\; (X-x)[1,2]_{yz} + (Y-y)[1,2]_{zx} + (Z-z)[1,2]_{xy}.$$

If R be the radius of the above sphere, then

$$R^2 = \Sigma\{-(1,1)[1,3]_{yz} + 3(1,2)[1,2]_{yz}\}^2/[1,2,3]^2$$

$$= \frac{(1,1)^2[1,3]^2 + 9(1,2)^2[1,2]^2 - 6(1,1)(1,2)[1,2\,|\,1,3].}{[1,2,3]^2}$$

If we take s_1 as the independent variable, then, because

$$(1,1)^{\frac{1}{2}} = \frac{ds_1}{dt} = 1 \quad \text{we have}$$

$$(1,1) = 1\,,\; D(1,1) = 0\,, \quad D^2(1,1) = 0.$$

Therefore $(1,2) = 0\,,\; (2,2) + (1,3) = 0.$

Again

$$\rho_1^2 = \frac{(1,1)^3}{[1,2]^2} \text{ therefore } [1,2]^2 = \frac{1}{\rho_1^2}$$

and $$(2,2) \equiv \frac{[1,2]^2 + (1,2)^2}{(1,1)} = \frac{1}{\rho_1^2}.$$

Also $$D(2,2) = 2(2,3)$$

therefore $$(2,3) = -\frac{1}{\rho_1^3}\,\frac{d\rho_1}{ds_1}$$

and $$[1,2\,|\,1,3] = (1,1)(2,3) - (1,2)(1,3) = (2,3) = -\frac{1}{\rho_1^3}\,\frac{d\rho_1}{ds_1}.$$

Again $$\rho_2 = \frac{[1,2]^2}{[1,2,3]} \qquad \text{therefore } [1,2,3] = \frac{1}{\rho_1^2\rho_2}$$

and because $$[1,3]^2[1,2]^2 - [1,2\,|\,1,3]^2 \equiv (1,1)[1,2,3]^2$$

therefore $$[1,3]^2 = \frac{1}{\rho_1^2\rho_2^2} + \frac{1}{\rho_1^4}\left(\frac{d\rho_1}{ds_1}\right)^2, \text{ and so on.*}$$

Evidently $R^2 = \dfrac{[1,3]^2}{[1,2,3]^2} = \rho_1^2 + \rho_2^2\left(\dfrac{d\rho_1}{ds_1}\right)^2$, a well-known formula.

* M. de Saint-Venant, *Journal de l'Ecole Polytechnique*, Cahier XXX, p. 64, gives a table of formulæ for three dimensional curves, in which he *virtually* calculates some of the parametric coefficients for s_1.

These formulae may be extended to a curve in space of any number of dimensions. We might, for instance, find an expression in the same way for the radius of a spheric of five pointic contact, to a curve in four dimensional space, in terms of ρ_1, ρ_2, ρ_3 and their derivatives with respect to s_1.*

In the same manner the equation of the osculating conicoid to a curve in three dimensional space can be written as

$$\begin{vmatrix} (X-x)^2, & \ldots\ldots, & (Y-y)(Z-z), & \ldots\ldots, & (X-x), & \ldots \\ O, & \ldots\ldots, & O, & \ldots\ldots, & Dx, & \ldots \\ 2\,Dx\,Dx, & \ldots\ldots, & 2Dy\,Dz, & \ldots\ldots, & D^2x, & \ldots \\ 6\,DxD^2x, & \ldots\ldots & 3DyD^2{}_3 + 3D^2yDz, & \ldots, & D^3x, & \ldots \\ 8\,DxD^3x + 6D^2xD^2x, & \ldots & 4DyD^3y + 6D^2yD^2y + 4D^3yDz, & \ldots & D^4x, & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 16DxD^7x + 56D^2xD^6x + \text{etc}, & & 8DyD^7x + 28D^2yD^6z + 56D^3yD^5z + \text{etc}, & \ldots & D^8x, & \ldots \end{vmatrix} = 0$$

If we multiply this by the transforming factor

$$\begin{vmatrix} [2,3]^2_{yz}, & [2,3]^2_{zx}, & [2,3]^2_{xy}, & 2[2,3]_{zx}[2,3]_{xy}, & 2[2,3]_{xy}[2,3]_{yz}, & 2[2,3]_{yz}[2,3]_{zx}, & O, & O, & O \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ [3,1]_{yz}[1,2]_{yz}, \ldots & & \ldots & [3,1]_{zx}[1,2]_{xy} + [3,1]_{xy}[1,2]_{zx} \ldots & & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ O, & O, & O, & O, & O, & O, & [2,3]_{yz}, & [2,3]_{zx}, & [2,3]_{xy} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{vmatrix}$$

and write

$$L_{23} \equiv (X-x)[2,3]_{yz} + (Y-y)[2,3]_{zx} + (Z-z)[2,3]_{xy}$$

$$L_{31} \equiv (X-x)[3,1]_{yz} + (Y-y)[3,1]_{zx} + (Z-z)[3,1]_{xy}$$

$$L_{12} \equiv (X-x)[1,2]_{yz} + (X-y)[1,2]_{zx} + (Z-z)[1,2]_{xy}$$

we get, by use of the reducing formulae

$$[m,n,p] \equiv [m,n]_{yz} D^p x + [m,n]_{zx} D^p y + [m,n]_{xy} D^p z$$

$$[m,n,p]\,[r,s,q] + [m,n,q]\,[r,s,p]$$

$$= 2[m,n]_{yz}[r,s]_{yz} D^p x D^q x + \ldots$$

$$+ \{[m,n]_{zx}[r,s]_{xy} + [m,n]_{xy}[r,s]_{zx}\}\{D^p{}_y D^q{}_z + D^p{}_z D^q{}_y\} + \ldots$$

---

* It can be shewn that, in this case, $R^2 = \rho^2 + (\rho'\sigma)^2 + \{(\rho'\sigma)' + \rho/\sigma\}^2(\rho/\sigma)^{\frac{2}{3}}\tau^2$, where ρ, σ, τ are the first three radii of curvature and the primes indicate differentiation with respect to the arc.

the following determinant equation

L^2_{23}	L^2_{31}	L^2_{12}
0	0	0
2 [1, 2, 3]^2	0	0
0	0	0
0	6 [1, 2, 3]^2	0
10 [1, 2, 3] [2, 3, 4]	0	0
12 [1, 2, 4] [2, 3, 5]	−30 [1, 2, 3] [1, 3, 4]	20 [1, 2, 3]^2
14 [1, 2, 3] [2, 3, 6]	−42 [1, 2, 3] [1, 3, 5]	70 [1, 2, 3] [1, 2, 4]
16 [1, 2, 3] [2, 3, 7]+70 [2, 3, 4]^2,	−56 [1, 2, 3] [1, 3, 6]+70 [1, 3, 4]^2,	112 [1, 2, 3] [1, 2, 5]+70 [1, 2, 4]^2

$L_{23}\,L_{31}$	$L_{31}\,L_{12}$
0	0
0	0
3 [1, 2, 3]^2	0
0	0
−5 [1, 2, 3] [1, 3, 4]	10 [1, 2, 3]^2
−6 [1, 2, 3] [1, 3, 5]+15 [1, 2, 3] [2, 3, 4]	15 [1, 2, 3] [1, 2, 4]
−7 [1, 2, 3] [1, 3, 6]+21 [1, 2, 3] [2, 3, 5]	21 [1, 2, 3] [1, 2, 5]−35 [1, 2, 3] [1, 3, 4]
−8 [1, 2, 3] [1, 3, 7]+28 [1, 2, 3] [2, 3, 6]−70 [1, 3, 4] [2, 3, 4],	28 [1, 2, 3] [1, 2, 6]−56 [1, 2, 3] [1, 3, 5]−70 [1, 2, 4] [1, 3, 4],

$$
\left|\begin{array}{llll}
L_{12}L_{23} & L_{23} & L_{31} & L_{12} \\
0 & [1,2,3] & 0 & 0 \\
0 & 0 & [1,2,3] & 0 \\
0 & 0 & 0 & [1,2,3] \\
4[1,2,3]^2 & [2,3,4], & -[1,3,4], & [1,2,4] \\
5[1,2,3][1,2,4] & [2,3,5], & -[1,3,5], & [1,2,5] \\
6[1,2,3][1,2,5] & [2,3,6], & -[1,3,6], & [1,2,6] \\
7[1,2,3][1,2,6]+35[1,2,3][2,3,4] & [2,3,7], & -[1,3,7], & [1,2,7] \\
8[1,2,3][1,2,7]+56[1,2,3][2,3,5]+70[1,2,4][2,3,4], & [2,3,8], & -[1,3,8], & [1,2,8]
\end{array}\right|=0
$$

of which a first simplification is

$$
\left|\begin{array}{lll}
L^2_{23}-2L_{31}[1,2,3] & L^2_{31} & L^2_{12} \\
2[1,3,4] & 6[1,2,3] & 0 \\
10[2,3,4]+2[1,3,5] & 0, & 0 \\
12[2,3,5]+2[1,2,6] & -30[1,3,4] & 20[1,2,3] \\
14[2,3,6]+2[1,3,7] & -45[1,3,5] & 70[1,2,4] \\
16[1,2,3][2,3,7]+70[2,3,4]^2+2[1,2,3][1,3,8], & -56[1,2,3][1,3,6]+70[1,3,4]^2, & 112[1,2,3][1,2,5]+70[1,2,4]^2,
\end{array}\right.
$$

$L_{14}\, L_{23}$	$L_{12}\, L_{34}$
$4\,[1, 2, 3]$	0
$5\,[1, 2, 4]$	$10\,[1, 2, 3]$
$6\,[1, 2, 5]$	$15\,[1, 2, 4]$
$7\,[1, 2, 6]+35\,[2, 3, 4]$	$21\,[1, 2, 5]-35\,[1, 3, 4]$
$8\,[1, 2, 3]\,[1, 2, 7]+56\,[1, 2, 3]\,[2, 3, 5]+70\,[1, 2, 4]\,[2, 3, 4],$	$28\,[1, 2, 3]\,[1, 2, 6]-56\,[1, 2, 3]\,[1, 3, 5]+70\,[1, 2, 4]\,[1, 3, 4].$

$$
\left.\begin{array}{l}
L_{23}\, L_{34}-3\,[1, 2, 3]\, L_{12} \\
\quad -3\,[1, 2, 4] \\
\quad -5\,[1, 3, 4]-3\,[1, 2, 5] \\
\quad -6\,[1, 3, 5]-3\,[1, 2, 6]+15\,[2, 3, 4] \\
\quad -7\,[1, 3, 6]-3\,[1, 2, 7]+21\,[2, 3, 5] \\
\quad -8\,[1, 3, 7]\,[1, 2, 3]-3\,[1, 2, 3]\,[1, 2, 8]-70\,[1, 3, 4]\,[2, 3, 4]+28\,[1, 2, 3]\,[2, 3, 6]
\end{array}\right| =0
$$

If we take the third intrinsic parameter s_3 to be the independent variable t, then because

$$\frac{ds_3}{dt}=[1, 2, 3]^{\frac{1}{6}}$$

we have

$$[1, 2, 3]=1 \quad D\,[1, 2, 3]=[1, 2, 4]=0$$

If we call $[1, 3, 4]=-I$ and $[2, 3, 4]=-J$

all the other parametric coefficients of class 3 can be calculated in terms of I and J, and their derivatives with respect to s_3.

The formulae for calculation of the parametric coefficients of class three are

$$\begin{aligned}
&[1, 2, n+1]=D\,[1, 2, n]-[1, 3, n]\\
&[2, 3, n+1]=D\,[2, 3, n]-[2, 4, n]\\
&[1, 3, n+1]=D\,[1, 3, n]-[2, 3, n]-[1, 4, n]
\end{aligned}$$

and

$$[1, 4, n]=\frac{[1, 2, 4]\,[1, 3, n]-[1, 3, 4]\,[1, 2, n]}{[1, 2, 3]}$$

$$[2, 4, n]=\frac{[1, 2, 4]\,[2, 3, n]-[2, 3, 4]\,[1, 2, n]}{[1, 2, 3]}$$

which are particular cases of the general formula

$$-[l, m, n]\,[1, 2, 3]^2=\begin{vmatrix} [2, 3, l], & [1, 3, l], & [1, 2, l] \\ [2, 3, m], & [1, 3, m], & [1, 2, m] \\ [2, 3, n], & [1, 3, n], & [1, 2, n] \end{vmatrix}$$

If the independent variable be the third intrinsic parameter s_3, then

$[1, 2, 3]=1, \quad [1, 2, 4]=0, \quad [1, 3, 4]=-I$
$[2, 3, 4]=-J$, therefore $[1, 4, n]=I\,[1, 2, n]$
and $[2, 4, n]=J\,[1, 2, n]$.

By the above formulae we obtain the following table of values of the parametric coefficients of class 3 :—

$[1, 2, 3]=1$	$[1, 2, 5]=I$	$[1, 2, 6]=2\,I'-J$
$[1, 2, 4]=0$	$[2, 3, 5]=J'$	$[2, 3, 6]=-J''-IJ$
$[1, 3, 4]=-I$	$[1, 3, 5]=-I'+J$	$[1, 3, 6]=-I''-I^2+2\,J'$
$[2, 3, 4]=-J$	$[1, 4, 5]=I^2$	$[1, 4, 6]=2\,II'-IJ$
	$[2, 4, 5]=IJ$	$[2, 4, 6]=2\,I'J-J^2$

$$\begin{aligned}
&[1, 2, 7]=3\,I''-3\,J'+I^2\\
&[2, 3, 7]=-J'''-3\,I'J-IJ'+J^2\\
&[1, 3, 7]=-I'''-4\,II'+3\,J''+2\,IJ\\
&[1, 4, 7]=3\,II''-3\,IJ'+I^3\\
&[2, 4, 7]=3\,I''J-3\,JJ'+I^2J
\end{aligned}$$

$[1, 2, 8] = 4\,I''' - 6\,J'' + 6\,II' - 2\,IJ$
$[2, 3, 8] = -J^{iv} - 6\,I''J - 4\,I'J' - IJ'' + 5\,JJ' - I^2J$
$[1, 3, 8] = -I^{iv} + 4\,J''' - 4\,I'^2 - 7\,II'' + 5\,I'J + 6\,IJ' - J^2 - I^3$
$[1, 4, 8] = 4\,I'''I - 6\,J''I + 6\,I^2I' - 2\,I^2J$
$[2, 4, 8] = 4\,I'''J - 6\,J''J + 6\,II'J - 2\,IJ^2$

and so on.

The equation of the osculating conicoid is simplified if we take s_3 as the independent variable and substitute the values of the parametric coefficients from the above table.

The foregoing part of the paper is intended as an introduction to the general methods and conceptions. In the next part the special case of parametric coefficients in two dimensions will be dealt with.

II. Application to Plane Curves.

1. Definitions and General Relations.

Suppose x, y are the co-ordinates of a point P of a plane curve defined by

$$x = F_1\,(t) \text{ and } y = F_2\,(t),$$

where F_1 and F_2 are given functions of the parameter t.

Then, if $D^n x$ and $D^n y$ be the n^{th} derivatives of x and y, with respect to t,

$$D^m x\, D^n x + D^m y\, D^n y \equiv (m, n)$$

and

$$D^m x\, D^n y - D^n x\, D^m y \equiv [m, n]$$

where (m, n) and $[m, n]$ are parametric coefficients of classes 1 and 2, respectively. We have $(m, n) = (n, m)$ and $[m, m] = 0$.

$$\text{Also } [m, n]\,[p, q] = \begin{vmatrix} (m, p), & (m, q) \\ (n, p), & (n, q) \end{vmatrix}$$

Whence $[m, n]\,[1, 2] = (1, m)\,(2, n) - (1, n)\,(2, m)$ and $[m, n]^2 = (m, m)\,(n, n) - (m, n)^2$.

If we multiply together the matrices

$$\begin{vmatrix} D^l x, & D^l y \\ D^m x, & D^m y \\ D^n x, & D^n y \end{vmatrix} \text{ and } \begin{vmatrix} D^p x, & D^p y \\ D^q x, & D^q y \\ D^r x, & D^r y \end{vmatrix} \text{ we get}$$

$$\begin{vmatrix} (l, p), & (l, q), & (l, r) \\ (m, p), & (m, q), & (m, r) \\ (n, p), & (n, q), & (n, r) \end{vmatrix} = 0$$

from which, putting $m = 1$, $n = 2$, $q = 1$, $r = 2$

we get

$$(l, p)\,[1, 2]^2 = (1, l)\,(1, p)\,(2, 2) + (2, l)\,(2, p)\,(1, 1) \\ -\{(1, l)\,(2, p) + (2, l)\,(1, p)\}\,(1, 2).$$

Also, by expanding the determinant, we have

$$(l, p)\,[m, n]\,[q, r] + (m, p)\,[n, l]\,[q, r] + (n, p)\,[l, m]\,[q, r] = 0$$

or, $\quad (l, p)\,[m, n] + (m, p)\,[n, l] + (n, p)\,[l, m] = 0$

whence $\quad (l, p)\,[1, 2] = [1, l]\,(2, p) - [2, l]\,(1, p)$

and $\quad (1, 1)\,[m, n] = [1, n]\,(1, m) - [1, m]\,(1, n)$

If we multiply together the matrices

$$\begin{vmatrix} D^l x, & -D^l y \\ D^m x, & -D^m y \\ D^n y, & D^n x \end{vmatrix} \text{ and } \begin{vmatrix} D^p y, & D^p x \\ D^q y, & D^q x \\ D^r y, & D^r x \end{vmatrix}, \text{ we get}$$

$$\begin{vmatrix} [l, p], & [l, q], & [l, r] \\ [m, p], & [m, q], & [m, n] \\ (n, p), & (n, q), & (n, r) \end{vmatrix} = 0$$

and if we multiply together the matrices

$$\begin{vmatrix} -D^l y, & D^l x \\ D^m x, & D^m y \\ D^n x, & D^n y \end{vmatrix} \text{ and } \begin{vmatrix} D^p x, & D^p y \\ D^q x, & D^q y \\ D^r x, & D^r y \end{vmatrix}, \text{ we get}$$

$$\begin{vmatrix} [l, p], & [l, q], & [l, r] \\ (m, p), & (m, q), & (m, n) \\ (n, p), & (n, q), & (n, r) \end{vmatrix} = 0 .$$

where, if we put $m=1$, $n=2$, $q=1$, $r=2$, we get

$$[l, p]\,[1, 2]^2 = \{[1, l]\,(2, p) + [2, l]\,(1, p)\}\,(1, 2) \\ -[1, l]\,(1, p)\,(2, 2) - [2, l]\,(2, p)\,(1, 1).$$

Also, by expanding the determinant, we have

$$[l, p]\,[m, n]\,[q, r] + [l, q]\,[m, n]\,[r, p] + [l, r]\,[m, n]\,[p, q] = 0$$

or $\quad [l, p]\,[q, r] + [l, q]\,[r, p] + [l, r]\,[p, q] = 0$

whence, $\quad [l, p]\,[1, 2] = [1, l]\,[2, p] - [1, p]\,[2, l]$

If we multiply together the matrices

$$\begin{vmatrix} -D^l y, & D^l x \\ D^m x, & D^m y \\ D^n x, & D^n y \end{vmatrix} \text{ and } \begin{vmatrix} -D^p y, & D^p x \\ D^q x, & D^q y \\ D^r x, & D^r y \end{vmatrix} \text{ we get}$$

$$\begin{vmatrix} -(l, p), & [l, q], & [l, r] \\ [m, p], & (m, q), & (m, r) \\ [n, p], & (n, q), & (n, r) \end{vmatrix} = 0$$

where if we put $m=1$, $n=2$, $q=1$, $r=2$, we get

$$(l,\ p)\ [1,\ 2]^2=[1,\ l]\ [1,\ p]\ (2,\ 2)+[2,\ l]\ [2,\ p]\ (1,\ 1)$$
$$-\{[1,\ l]\ [2,\ p]+[2,\ l]\ [1,\ p]\}\ (1,\ 2).$$

Finally, if we multiply together the matrices

$$\begin{vmatrix} -D^l y, & D^l x \\ D^m x, & D^m y \\ D^n x, & D^n y \end{vmatrix} \text{ and } \begin{vmatrix} D^p x, & D^p y \\ -D^q y, & D^q x \\ -D^r y, & D^r x \end{vmatrix} \text{, we get}$$

$$\begin{vmatrix} -[l,\ p], & (l,\ q), & (l,\ r) \\ (m,\ p), & [m,\ q], & [m,\ r] \\ (m,\ p), & [n,\ q], & [n,\ r] \end{vmatrix}=0$$

where if we put $m=1$, $n=2$, $q=1$, $r=2$, we get

$$[l,\ p]\ [1,\ 2]=(1,\ l)\ (2,\ p)-(1,\ p)\ (2,\ l)$$

a relation otherwise evident.

2. Working Formulae.

All the other parametric coefficients can be easily calculated, if we know $(1,\ 1)$ and $(2,\ 2)$, and therefore $(1,\ 2)$, $(1,\ 3)$, $[1,\ 2]$, $(2,\ 3)$ and $[2,\ 3]$.

For $D(1,\ 1)=2(1,\ 2)$, $[1,\ 2]^2=(1,\ 1)\ (2,\ 2)-(1,\ 2)^2$,

$D(2,\ 2)=2(2,\ 3)$, $D(1,\ 2)=(1,\ 3)+(2,\ 2)$

and $[1,\ 2]\ [2,\ 3]=(1,\ 2)\ (2,\ 3)-(1,\ 3)\ (2,\ 2)$.

The following set of formulae, which we may call the first set, will be found useful for general purposes.

Suppose we have calculated $(1,\ n)$ and $[1,\ n]$ for all values of n from 1 to n; then we can calculate $(r,\ s)$ and $[r,\ s]$, for values of r and s not exceeding n, from

$$(r,\ s)\ (1,\ 1)=(1,\ r)\ (1,\ s)+[1,\ r]\ [1,\ s]$$

and
$$[r,\ s]\ (1,\ 1)=(1,\ r)\ [1,\ s]-(1,\ s)\ [1,\ r].$$

We can next calculate $(1,\ n+1)$ and $[1,\ n+1]$ from

$$(1,\ n+l)=D(1,\ n)-(2,\ n)$$
$$[1,\ n+l]=D[1,\ n]-[2,\ n]$$

where
$$(2,\ n)=\{(1,\ 2)(1,\ n)+[1,\ 2]\ [1,\ n]\}/(1,\ 1)$$

and
$$[2,\ n]=\{(1,\ 2)\ [1,\ n]-[1,\ 2]\ (1,\ n)\}/(1,\ 1).$$

The following set of formulae, which we may call the second set, are also useful, and specially so, in certain cases.

Suppose we have calculated $[1, n]$ and $[2, n]$ for all values of n from 1 to n. Then we can calculate $[r, s]$ for values of r and s not exceeding n, from

$$[r, s]=\{[1, r]\,[2, s]-[2, r]\,[1, s]\}/[1, 2].$$

We can next calculate $[1, n+1]$ and $[2, n+1]$ from

$$[1, n+1]=D[1, n]-[2, n]$$

and

$$[2, n+1]=D[2, n]-[3, n]$$

where

$$[3, n]=\{[1, 3]\,[2, n]-[2, 3]\,[1, n]\}/[1, 2].$$

Subsequently we can calculate $(1, n)$ $(2, n)$ and (r, s) from

$$(1, n)\,[1, 2]=(1, 2)\,[1, n]-(1, 1)\,[2, n]$$

$$(2, n)\,[1, 2]=(2, 2)\,[1, n]-(1, 2)\,[2, n]$$

and

$$[1, 2]^2(r, s)=[1, r]\,[1, s]\,(2, 2)+[2, r]\,[2, s]\,(1, 1)$$
$$-\{[1, r]\,[2, s]+[1, s]\,[2, r]\}\,(1, 2).$$

The following set of formulae, which we may call the third set, may also be used.

Suppose we have calculated $(1, n)$ and $(2, n)$ for all values of n from 1 to n. Then we can calculate (r, s) and $[r, s]$ for values of r and s not exceeding n, from

$$[1, 2]^2(r, s)=(1, r)(1, s)(2, 2)+(2, r)(2, s)(1, 1)$$
$$-\{(1, r)(2, s)+(1, s)(2, r)\}(1, 2)$$

and

$$[r, s]=\{(1, r)(2, s)-(1, s)(2, r)\}/[1, 2].$$

We can next calculate $(1, n+1)$ and $(2, n+1)$, from

$$(1, n+1)=D(1, n)-(2, n)$$

and

$$(2, n+1)=D(2, n)-(3, n)$$

where

$$(3, n)=\frac{[1, 3](2, n)-[2, 3](1, n)}{[1, 2]}$$

3. When the first intrinsic parameter (s_1) is the independent variable t.

In this case, since $(1, 1)^{\frac{1}{2}}=\dfrac{ds_1}{dt}=1$, we have $(1, 1)=1$,

$(1, 2)=0$ for $D(1, 1)=2(1, 2)$.

Also $[1, 2]=r$, where $r=\dfrac{[1, 2]}{(1, 1)^{\frac{3}{2}}}=\dfrac{1}{\rho}$

Now, if we use the first set of formulae, we get

$$(2, n)=\{(1, 2)(1, n)+[1, 2]\,[1, n]\}/(1, 1)=r[1, n]$$

$$[2, n]=\{(1, 2)\,[1, n]-[1, 2]\,(1, n)\}/(1, 1)=-r(1, n).$$

Therefore

$$(2, 2)=r^2,\ (1, 3)=D(1, 2)-(2, 2)=-r^2$$

$$(2, 3)=rr',\ \text{for } D(2, 2)=2(2, 3)$$

$$[1, 3]=D[1, 2]=r',\ [2, 3]=-r(1, 3)=r^3$$

$$(3, n)=\{(1, 3)(1, n)+[1, 3]\,[1, n]\}/(1, 1)=-r^2(1, n)+r'[1, n]$$

$$[3, n]=\{(1, 3)\,[1, n]-[1, 3]\,(1, n)\}/(1, 1)=-r^2[1, n]-r'(1, n)$$

so that we can at once write down the series $(2, n)$, $[2, n]$, $(3, n)$, $[3, n]$ if we calculate the series $(1, n)$ and $[1, n]$.

The values of the series $(1, n)$, $[1, n]$, calculated from the first set of formulae, are

$$(1, 1)=1,\ [1, 2]=r,\ (1, 2)=0$$

$$(1, 3)=-r^2,\ [1, 3]=r'$$

$$(1, 4)=-3rr',\ [1, 4]=r''-r^3$$

$$(1, 5)=-4rr''-3r'^2+r^4,\ [1, 5]=r'''-6r^2r'$$

$$(1, 6)=-5rr'''-10r'r''+10r^3r'$$

$$[1, 6]=r^{iv}-10r^2r''-15rr'^2+r^5$$

$$(1, 7)=-6rr^{iv}-15r'r'''-10r''^2+20r^3r''+45r^2r'^2-r^6$$

$$[1, 7]=r^{v}-15r^2r'''-60rr'r''+15r^4r'-15r'^3$$

$$(1, 8)=-7rr^{v}-21r'r^{iv}-35r''r'''+35r^3r'''+210r^2r'r''+105rr'^3-21r^5r'$$

$$[1, 8]=r^{vi}-21r^2r^{iv}-105rr'r'''-70rr''^2-105r'^2r''+35r^4r''+105r^3r'^2-r^7$$

$$(1, 9)=-8rr^{vi}-28r'r^{v}-56r''r^{iv}+56r^3r^{iv}-35r'''^2+420r^2r'r'''+280r^2r''^2+840rr'^2r''-56r^5r''+105r'^4-210r^4r'^2+r^8$$

$$[1, 9]=r^{vii}-28r^2r^{v}-168rr'r^{iv}-280rr''r'''-210r'^2r'''-210r'r''^2+70r^4r'''+560r^3r'r''+420r^2r'^3-28r^6r',$$ and so on.

4. When the second intrinsic parameter s_2 is taken as the independent variable t.

In this case, since

$$[1, 2]^{\frac{1}{3}}=\frac{ds_2}{dt}=1,$$ we have $[1, 2]=1$ and $[1, 3]=D[1, 2]=0$.

Also, since $(1,\ 1)^{\frac{3}{2}}/[1,\ 2]=\rho$, where ρ is radius of curvature, we have $(1,\ 1)=\rho^{\frac{2}{3}}$.

Again, since $\{3[1,2]\,(1,2)-[1,3]\,(1,1)\}/3[1,2]^2=\tan\delta$, where δ is the angle of aberrancy (*vide*—A General Theory of Osculating Conics, Second Paper), we have $(1,\ 2)=\tan\delta$, and because $(1,\ 1)\ (2,\ 2)=[1,\ 2]^2+(1,\ 2)^2$, we have $(2,\ 2)=(\tan^2\delta+1)\rho^{-\frac{2}{3}}$ $=\sec^2\delta\,\rho^{-\frac{2}{3}}$.

Also because $\{3[1,\ 2][1,\ 4]-5[1,\ 3]^2+12[1,\ 2][2,\ 3]\}/9[1,\ 2]^{\frac{8}{3}}$ $=(ab)^{-\frac{2}{3}}$, where $a,\ b$ are the semiaxes of the osculating conic (*vide* A General Theory of Osculating Conics, Second Paper), we have

$$3[1,\ 4]+12[2,\ 3]=9(ab)^{-\frac{2}{3}}$$

$$\text{But } [1,\ 4]+[2,\ 3]=D[1,\ 3]=0$$

$$\text{therefore } [2,\ 3]=-[1,\ 4]=(ab)^{-\frac{2}{3}}=I \text{ suppose}$$

If we use the second set of formulae we have

$$[3,\ n]=\{[1,\ 3]\ [2,\ n]-[2,\ 3]\ [1,\ n]\}/[1,\ 2]=-I[1,\ n]$$

Therefore, starting from $[1,2]=1,[1,3]=0,[2,3]=I,[1,4]=-I$, we can calculate all the parametric coefficients of class 2.

The parametric coefficients of class 1 are then determined from

$$(1,\ n)=\{(1,\ 2)[1,\ n]-(1,\ 1)\ [2,\ n]\}/[1,\ 2]$$

$$(2,\ n)=\{(2,\ 2)\ [1,\ n]-(1,\ 2)[2,\ n]\}/[1,\ 2]$$

and

$$[1,\ 2]^2(r,\ s)=[1,\ r]\ [1,\ s]\ (2,\ 2)+[2,\ r]\ [2,\ s]\ (1,\ 1)$$
$$-\{[1,\ r]\ [2,\ s]+[1,\ s]\ [2,\ r]\ (1,\ 2)$$

which give

$$(1,\ n)=[1,\ n]\tan\delta-[2,\ n]\rho^{\frac{2}{3}}$$

$$(2,\ n)=[1,\ n]\sec^2\delta\,\rho^{-\frac{2}{3}}-[2,\ n]\tan\delta$$

and

$$(r,\ s)=[1,\ r]\ [1,\ s]\sec^2\delta\,\rho^{-\frac{2}{3}}-[2,\ r]\ [2,\ s]\ \rho^{\frac{2}{3}}$$
$$-\{[1,\ r]\ [2,\ s]+[1,\ s]\ [2,\ r]\}\tan\delta.$$

The values of the series $[1,\ n]$, $[2,\ n]$, calculated from the second set of formulae and expressed in terms of I and its derivatives with respect to s_2, are given in the following table which may be easily extended as far as one wishes. The series $[3,\ n]$ is at once obtained from the formula

$$[3,\ n]=-I\ [1,\ n]$$

and the series $[r, n]$ can be calculated from

$$[r, n] = \{[1, r]\,[2, n] - [2, r]\,[1, n]\}/[1, 2].$$

$[1, 2] = 1,\ [1, 3] = 0,\ [2, 3] = I$
$[1, 4] = -I,\ [2, 4] = I'$
$[1, 5] = -2I',\ [2, 5] = I'' - I^2$
$[1, 6] = -3I'' + I^3,\ [2, 6] = I''' - 4II'$
$[1, 7] = -4I''' + 6II',\ [2, 7] = I^{iv} - 7II'' - 4I'^2 + I^3$
$[1, 8] = -5I^{iv} + 13II'' + 10I'^2 - I^3$
$[2, 8] = I^{v} - 11II''' - 15I'I'' + 9I^2I'$
$[1, 9] = -6I^{v} + 24II''' + 48I'I'' - 12I^2I'$
$[2, 9] = I^{vi} - 16II^{iv} - 26I'I''' - 15I''^2 + 22I^2I'' + 28II'^2 - I^4$

and so on.

Note.—Calculations of the parametric coefficients for the systems (r, θ) and (s, ψ) are given in paper No. 6, mentioned in the Introduction. The general conception of parametric coefficients in two dimensions first arose in the above-mentioned paper although the conception of intrinsic parameters has been first introduced in the present paper.

5. Expressions for the length of a Chord.

Let P_0 and P_1, be two points on the curve corresponding to values o and t of the parameter. Let the co-ordinates of P_0 and P_1 be (x_0, y_0) and (x_1, y_1).
Then, evidently

$$(x_1 - x_0)\,Dx + (y_1 - y_0)\,Dy$$
$$= (1, 1)\,t + (1, 2)\,t^2/2! + (1, 3)\,t^3/3! + (1, 4)\,t^4/4! + \text{etc.},$$
$$\text{and } (y_1 - y_0)\,Dx - (x_1 - x_0)\,Dy$$
$$= [1, 2]\,t^2/2! + [1, 3]\,t^3/3! + [1, 4]\,t^4/4! + \text{etc.}$$

Similarly, if (x_1', y_1') and (x_0, y_0) be the co-ordinates of two points P'_1 and P_0 on the curve corresponding to values $-t$ and o of the parameter, then

$$(x_1' - x_0)\,Dx + (y_1' - y_0)\,Dy$$
$$= -(1, 1)\,t + (1, 2)\,t^2/2! - (1, 3)\,t^3/3! + (1, 4)\,t^4/4! - \text{etc.}$$

and

$$(y_1' - y_0)\,Dx - (x_1' - x_0)\,Dy$$
$$= [1, 2]\,t^2/2! - [1, 3]\,t^3/3! + [1, 4]\,t^4/4! - \text{etc.}$$

Therefore

$$\tfrac{1}{2}\{(x_1 - x_1')\,Dx + (y_1 - y_1')\,Dy\}$$
$$= (1, 1)\,t + (1, 3)\,t^3/3! + (1, 5)\,t^5/5! + \text{etc.}$$

and

$$\tfrac{1}{2}\{(y_1 - y_1')\,Dx - (x_1 - x_1')\,Dy\}$$
$$= [1, 3]\,t^3/3! + [1, 5]\,t^5/5! + \text{etc.}$$

If L be the length of the chord $P_1\ P_1'$ then

$$(1,\ 1)\ L^2=\{(x_1-x_1')\ Dx+(y_1-y_1')\ Dy\}^2$$
$$+\{(y_1-y_1')\ Dx-(x_1-x_1')\ Dy\}^2$$

whence, after simplications, we have

$$L^2/4\ t^2=(1,\ 1)+2(1,\ 3)\ t^2/3!+2\ (1,\ 5)\ t^4/5!+(3,\ 3)\ t^4/3!\ 3!$$
$$+2\ (1,\ 7)\ t^6/7!+2\ (3,\ 5)\ t^6/3!\ 5!+2\ (1,\ 9)\ t^8/9!$$
$$+2(3,\ 7)\ t^8/3!\ 7!+(5,\ 5)\ t^8/5!\ 5!+\text{etc.}$$

Whence again, after extraction of square root and simplications, we have

$$L/2\ t\ (1,\ 1)^{\frac{1}{2}}=1+\frac{t^2}{(1,\ 1)}\ \frac{(1,\ 3)}{3!}+\frac{t^4}{(1,\ 1)^2}\left\{\frac{(1,\ 5)\ (1,\ 1)}{5!}+\frac{1}{2}\frac{[1,\ 3]^2}{3!\,3!}\right\}$$
$$+\frac{t^6}{(1,\ 1)^3}\left\{\frac{(1,\ 7)\ (1,\ 1)^2}{7!}+\frac{[1,\ 3]\ [1,\ 5]\ (1,\ 1)}{3!\ 5!}-\frac{1}{2}\frac{(1,\ 3)\ [1,\ 3]^2}{3!\ 3!\ 3!}\right\}$$
$$+\frac{t^8}{(1,\ 1)^4}\left\{\frac{(1,\ 9)\ (1,\ 1)^3}{9!}+\frac{1}{2}\ \frac{(1,\ 1)^2\ [1,\ 5]^2}{5!\ 5!}-\frac{1}{2}\ \frac{(1,\ 5)\ [1,\ 3]^2\ (1,\ 1)}{3!\ 5!\ 3!}\right.$$
$$-\frac{(1,\ 3)\ [1,\ 3]\ [1,\ 5]\ (1,\ 1)}{5!\ 3!\ 3!}+\frac{[1,\ 3]\ [1,\ 7]\ (1,\ 1)^2}{3!\ 7!}+\frac{1}{2}\ \frac{(1,\ 3)^2\ [1,\ 3]^2}{3!\ 3!\ 3!\ 3!}$$
$$\left.-\frac{1}{8}\ \frac{[1,\ 3]^4}{3!\ 3!\ 3!\ 3!}\right\}+\text{etc.}$$

If the independent variable (t) be the second intrinsic parameter s_2, then $[1,\ 3]=0$, therefore the expression for L reduces to

$$L/2\ t\ (1,\ 1)^{\frac{1}{2}}=1+\frac{t^2}{(1,\ 1)}\ \frac{(1,\ 3)}{3!}+\frac{t^4}{(1,\ 1)}\ \frac{(1,\ 5)}{5!}+\frac{t^6}{(1,\ 1)}\ \frac{(1,\ 7)}{7!}+\text{\&c.}$$

or

$$L=2\ s_2\ \rho^{\frac{1}{3}}\left\{1-\frac{s_2^{\ 2}}{3!}\ I-\frac{s_2^{\ 4}}{5!}\ \frac{2\ I'\ \tan\delta-\rho^{\frac{2}{3}}\ (I''-I^2)}{\rho^{\frac{2}{3}}}+\text{etc.}\right\}$$

where $2\ s_2$ is the length of the second intrinsic parameter from P_1' to P_1, the initial point P_0 from which s_2 is measured being so situated as to bisect $2s_2$.

If the independent variable be the first intrinsic parameter s_1, namely, arc-length, then the expression for L becomes

$$L/2s_1=1-s_1^{\ 2}\ r^2/6+s_1^{\ 4}\ (3\ r^4-4\ r'^2-12\ rr'')/360+\text{etc.}$$

If we write s for the entire arc $P_1'P_1$ which is $2s_1$, we have

$$L=s-s^3r^2/24+s^5\ (3r^4-4r'^2-12rr'')/5760+\text{etc.}$$

If we shift the origin of s to any arbitary point on the arc so that the arc distances to P_1' and P_1 are s_1' and s_1, then

$$s=s_1-s_1'$$

and the arc distance of the old origin from the new is $(s_1+s_1')/2$. Therefore

$$L=(s_1-s_1')-\frac{(s_1-s_1')^3}{24}\left\{r^2+\frac{s_1+s_1'}{2}D\,(r^2)+\left(\frac{s_1+s_1'}{2}\right)^2 D^2\,(r^2)/2\right.$$

$$\left.+\left(\frac{s_1+s_1'}{2}\right)^3 D^3\,(r^2)/3\,!+\&c.\right\}+\frac{(s_1-s_1')^5}{5760}\left\{(3\,r^4-4\,r'^2-12\,rr'')\right.$$

$$\left.+\frac{s_1+s_1'}{2}\,D\,(3r^4-4\,r'^2-12\,rr'')+\text{etc.}\right\}+\text{etc.}$$

Note.—The expressions for L given here are interesting. The expression for L in terms of the arc is calculated in some text books (*vide* Calcul Differentiel par J. Bertrand) to a few terms, but the method is less general.

6. The Osculating Cubic.

Let x, y be a given point on a curve and X, Y another point so that the value of the second intrinsic parameter from the first to the second point is s_2. Then, if we write

$$L_1\equiv(Y-y)\ \frac{dx}{ds_2}-(X-x)\ \frac{dy}{ds_2}$$

and

$$L_2\equiv(Y-y)\ \frac{d^2x}{ds^2_2}-(X-x)\ \frac{d^2y}{ds^2_2}$$

we have

$$L_1\equiv[1,\,2]\,\frac{s^2_2}{2\,!}+[1,\,3]\,\frac{s^3_2}{3\,!}+[1,\,4]\,\frac{s^4_2}{4\,!}+\&c.$$

$$L_2\equiv-[1,\,2]s_2+[2,\,3]\,\frac{s^3_2}{3\,!}+[2,\,4]\,\frac{s^4_2}{4\,!}+\&c.$$

or,

$$L_1=s^2_2/2\,!-Is^4_2/4\,!-2I's^5_2/5\,!-(3I''-I^2)s^6_2/6\,!$$
$$-(4I'''-6II')s_2^7/7\,!-(5I^{iv}-13II''-10I'^2+I^3)s^8_2/8\,!$$
$$-(6I^{v}-24II'''-48I'I''+12I^2I')s^9_2/9\,!-\&c.$$

and

$$L_2=-s_2+Is_2^3/3\,!+I's_2^4/4\,!+(I''-I^2)s_2^5/5\,!$$
$$+(I'''-4II')s^6_2/6\,!+(I^{iv}-7II''-4I'^2+I^3)s_2^7/7\,!$$
$$+(I^{v}-11II'''-15I'I''+9I^2I')\,s^8_2/8\,!+(I^{vi}-16II^{iv}$$
$$-26I'I'''-15I''^2+22I^2I''+28II'^2-I^4)s_2^9/9\,!+\&c.$$

Whence,

$$L_1^2 = 6s_2^4/4! - 30Is_2^6/6! - 84I's_2^7/7!$$
$$-(168I'' - 126I^3)s_2^8/8! - (288I''' - 936II')s_2^9/9! + \&c.$$
$$L_1^3 = 90s_2^6/6! - 1260Is_2^8/8! - 4536I's_2^9/9! + \&c.$$
$$L_2^2 = 2s_2^2/2! - 8Is_2^4/4! - 10I's_2^5/5! - (12I'' - 32I^2)s_2^6/6!$$
$$-(14I''' - 126II')s_2^7/7! - (16I^{iv} - 224II'' - 134I'^2 +$$
$$128I^3)s_2^8/8!$$
$$-(18I^{v} - 366II''' - 522I'I'' + 1086I^2I')s_2^9/9! - \&c.$$
$$S \equiv L_2^2 - 2L_1 + IL_1^2 = -6I's_2^5/5! - 6I''s_2^6/6! - (6I''' -$$
$$30II')s_2^7/7!$$
$$-(6I^{iv} - 30II'' - 114I'^2)s_2^8/8! - (6I^{v} - 426I'I'' - 30II''' +$$
$$126I^2I')s_2^9/9! + \&c.$$

So that $S=O$ is the equation of the osculating conic and $I=O$, $I'=O$, are respectively the differential equations of the parabola and general conic respectively.

$$L_1S = -126I's_2^7/7! - 168I''s_2^8/8! - (216I''' - 1836II')$$
$$s_2^9/9! - \&c.$$
$$L_2S = 36I's_2^6/6! + 42I''s_2^7/7! + (48I''' - 576II')s_2^8/8!$$
$$+(54I^{iv} - 774II'' - 1782I'^2)s_2^9/9! + \&c.$$

Therefore $\qquad W_1 \equiv -5I''L_1S + 6I'^2L_1^3 - 15I'L_2S$

$$= -(720I'I''' - 840I''^2 - 1080II'^2)s_2^8/8! - (810I'I^{iv} -$$
$$1080I''I''' - 2430II'I'' + 486I'^3)s_2^9/9! + \&c.$$

and $\qquad W_2 \equiv 7I'(-15S + 3I'L_1^2L_2 - I''L_1^3) + 5(I''' + 9II')L_1S$

$$= (630I'I^{iv} - 840I''I''' - 1890II'I'' + 10962I'^3)s_2^8/8!$$
$$+(630I'I^{iv} - 1080I'''^2 - 3690II'I''' + 34650I'^2I'' + 3240I^2I'^2)$$
$$s_2^9/9! + \&c.$$

Hence $\quad W_1(630I'I^{iv} - 840I''I''' - 1890II'I'' + 10962I'^3)$

$$+ W_2(720I'I''' - 840I''^2 - 1080II'^2) = 0$$

or

$$7(15I'I^{iv} - 20I''I''' - 45II'I'' + 261I'^3)W_1$$
$$+ 20(6I'I''' - 7I''^2 - 9II'^2)W_2 = 0$$

is the equation of the osculating cubic.

The differential equation of the general cubic is then

$$21(15I'I^{iv} - 20I''I''' - 45II'I'' + 261I'^3)(15I'I^{iv} - 20I''I'''$$
$$- 45II'I'' + 9I'^3)$$
$$= 100\,(6I'I''' - 7I''^2 - 9II'^2)\,(7I'I^{v} - 12I'''^2 - 41II'I''' + 385I'^2I''$$
$$+ 36I^2I'^2)$$

The above direct forms and methods of deduction of the equation of the osculating cubic and the general differential equation of the cubic are interesting.

For the Laguerre-Forsyth forms see "Projective Differential Geometry, etc." by Wilczynski, mentioned in the introduction.

A number of other applications of parametric coefficients occur in paper No. 6.

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